A damped pendulum forced with a constant torque

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The dynamics of a pendulum damped and forced with a constant torque is studied experimentally and theoretically. This simple device alllows to introduce and demonstrate some generical dynamical behaviors including the loss of equilibria (or saddle node bifurcation) and the homoclinic bifurcation. A qualitative analysis is developped to emphasize the role of damping and inertia.

I. INTRODUCTION

Bifurcation theory allows to classify qualitative changes that occurs in dynamical systems. The lost of equilibria is the most simple bifurcation. It occurs in many physical systems. A pending drop of water at the extremity of a faucet or a syringe provides a simple example of such a bifurcation. One can make a small drop that hangs under the action of its weight, the action of the pressure of the surrounding fluid (water in the faucet and air) and the action of surface tension at the interface between water and air. The volume of the drop can be increased until a critical volume is reached. It is impossible to form a drop with a volume higher than this critical one. Another example is given by the lost of equilibria of a solid on an inclined plane when the slope exceeds a critical one. From a mathematical view point, the lost of equilibria is associated with the lost of zeroes of a real function when they become complex.

In this paper we will focus on a mechanical example proposed by Andronov et al.[2]. It consists of a damped pendulum forced with a constant torque. It can be easily built and the physics involved in this device is very simple. However, its dynamics are rich and the system exhibit much more than a loss of equilibria. Such fundamental notions as hysteresis, bistability between equilibrium and periodic solutions, homoclinic bifurcation will also be demonstrated.

The aim of the paper is to explore the dynamics of the pendulum. Our main concern will be to make an introduction to bifurcations and complex dynamical behaviors from the experimental side. The emphasis will be made on qualitative analysis. We show that the dynamic depends only on two parameters and we will explore the parameter space.

The paper is organized as follows : the first part is devoted to the description of the experimental setup and its model, in the second part some behavior observed on experiments are described. In the third part, we present a qualitative analog of the pendulum. These three parts form the basis for an early introduction to qualitative dynamics, based on experimental data. In the fourth part, the system is explored analytically in more details.

II. EXPERIMENTAL SETUP AND EQUATION OF MOTION

A. Experimental setup

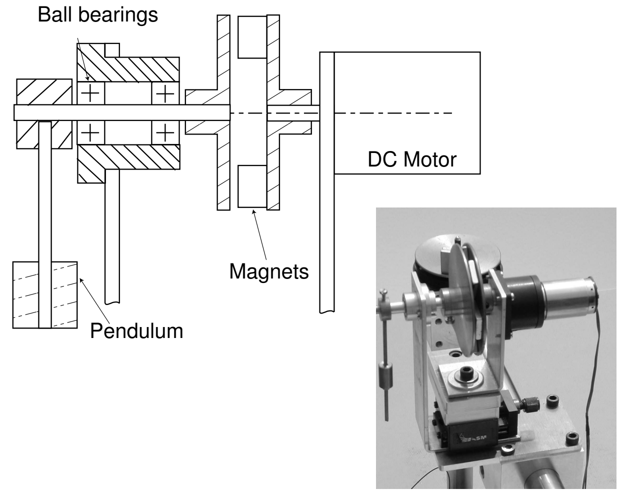

FIG. 1: Sketch and photography of the experimental setup.

A planar pendulum is forced by a constant torque (figure 1). The pendulum is linked to the driving shaft of a DC motor through a magnetic coupler. This device is made of two coaxial disks (radius 40 mm) facing each other. One of the disk (denoted M-disk) is attached to the drive shaft and has a few (8 in our case) neodym-iron-bore strong permanent magnets sticked on its surface. The other disk (denoted P-disk) made of an aluminum alloy is attached to the pendulum. When the driving shaft rotates, induced currents are generated in the P-disk and these current are opposing the changing of the location of the flux[?] . The rotation of the motor induces a magnetic viscous torque on the pendulum $T = \nu(\dot{\theta} - \Omega_m)$ where $\dot{\theta}$ is the angular velocity of the pendulum and Ω_m is the motor angular velocity. ν is the viscous torque per unit of rotation speed and depends on the strength of the magnets and the distance between the two disks. Moreover, the pendulum rotates on ball bearings. This leads to an additionnal frictional torque which is more difficult to model. However, we will consider that this frictional moment is much smaller than the frictional moment due to magnetic effects.

We use a voltage regulated power supply for the DC motor. The angular velocity of the motor is constant. The power of the motor is 7 Watts. The angular velocity that we finally get on the pendulum axis after reduction through a reductor and a driving belt is approximately 1 rps. It is easily changed by changing the supply voltage. The whole setup can be rotated by an angle α around an axis which is perpendicular to the plane formed by the gravity and the axis of rotation of the pendulum. Then the effective gravity is $g = g_0 \cos\alpha$. During the experiment, we will vary Ω_m and g.

We have deliberately chosen to describe qualitative experiments. We will only use a stop watch to measure the period of oscillation of the pendulum and the angular velocity and a protractor to measure the angle of deviation at equilibrium. This measure only aims to give some order of magnitude in order to help the reader to reproduce this experiment.

B. Equation of motion

The equation for the angular momentum of the pendulum is

$$I\ddot{\theta} + mgl\sin\theta = \nu(\Omega_M - \dot{\theta}) - \nu_a\dot{\theta}, \tag{1}$$

where I represents the total moment of inertia $I = I_D + ml^2$, I_D is the inertia of the disk, m is the mass of the pendulum, l is the distance between the pendulum's center of inertia and the axis of rotation of the pendulum, ν is the viscous torque per unit of angular velocity due to the magnetic friction. ν_a is a friction coefficient that measures the friction torque per unit of angular velocity exerted on the pendulum by the surrounding air and the ball bearings. ν_a will be neglected. The time is measured in units of the free oscillating period and the adimensionnalized time $\tau = (mgl/I)^{1/2}t$ will be used. The equation of motion reads

$$\theta'' + \beta\theta' + \sin\theta = \gamma, \tag{2}$$

where $'$ denotes the derivative with respect to τ, and

$$\beta = \frac{\nu}{\sqrt{mglI}}, \quad \gamma = \frac{\nu\Omega_m}{mgl}. \tag{3}$$

Equilibria θ_e are solutions of

$$\sin\theta = \gamma, \tag{4}$$

III. EXPERIMENTS

A. Order of magnitudes

We give an order of magnitude of β in our experiment. β is given by

$$\beta = \frac{\nu}{mgl}\sqrt{\frac{mgl}{I}} = \frac{a}{2\pi}\frac{2\pi}{T_0}. \tag{5}$$

a is the slope of the line $\sin\theta_e = f(\Omega_m)$ with Ω_m in rotation per second and T_0 is the period of oscillations of the undamped pendulum (in its linear regime). The period is measured with the stop watch and it is approximately 0.9 seconds. a is determined on figure 2 where the result of two series of measurements are reported. The angular velocities have been measured with a stop watch and the angle of tilt is measured with the protractor. For

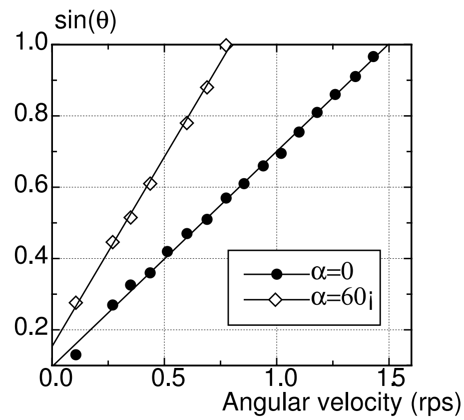

FIG. 2: Measure of the sinus of the angle of equilibrium vs the motor rotation speed for two different inclination angle of the pendulum.

the curve $\alpha = 0$, the axis of the pendulum is horizontal, $a_0 = 1.1(rps)^{-1}$ and $\beta = 1.2$. For the curve $\alpha = 60\,\text{deg}$, $a_{60} = 0.6(rps)^{-1}$ (the ratio between a_0 and a_{60} was supposed to be 2) and $\beta = 0.7$.

B. Dynamical experiments

Scenario 1 : $\alpha = 0$
When the motor doesn't work, the pendulum hangs with $\theta = 0$. When the motor rotates slowly, the equilibrium state of the pendulum is tilted. When the rotation speed increases, the equilibrium approaches $\theta = \pi/2$ and at a critical velocity of the motor Ω_c, the pendulum starts to rotate. This is the loss of equilibria.

We go back to a given velocity $\Omega_M < \Omega_c$ but not too small. The pendulum has a stable equilibrium position θ_s. One can perturb the pendulum *i.e.* one can release it, without initial velocity, from an angle θ_i. If the perturbation of the pendulum is in the direction opposite to the motor rotation ($\theta_i < \theta_s$), the pendulum goes back smoothly to its equilibrium (figure 3). If one perturbs the pendulum in the direction of the rotation, when the angle of perturbation is small, the pendulum goes back to equilibrium against the torque because the gravity is too strong. But if the initial angle is large enough ($\theta_i > \pi - \theta_s$)[5], the pendulum does not go against the torque but it moves in the direction of the torque.

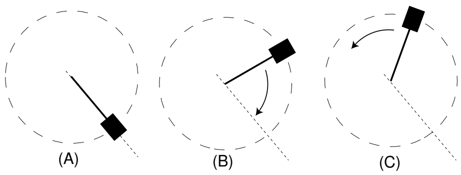

FIG. 3: Perturbation of the pendulum

If Ω_M is not too small, the pendulum starts to rotate and it reaches rapidly a periodic regime. If we stop by hand the pendulum and put it close to its equilibrium, it will stay there. There is bistability between the rotating periodic regime and the stationnary equilibrium. When the pendulum rotates, if we decrease Ω_M, at some point the pendulum will stop. We notice that when the pendulum stops it tries to reach some angle higher than $\pi/2$ and then stops and goes back to the stable equilibrium. In this regime if one perturbs the pendulum, it goes back to equilibrium. There is no more rotating behavior.

If one increases the torque without perturbing the pendulum, the angle of equilibrium increases and reaches $\pi/2$ when $\Omega = \Omega_c$. For Ω even slighly greater than Ω_c, the pendulum rotates fast and does not slow down close to the angle $\pi/2$. There is a discontinuity in the behavior of the pendulum, it jumps to a periodic regime. If one decreases the motor speed, the pendulum goes on rotating, even if the speed is $\Omega < \Omega_c$. The loss of equilibria is a transition with hysteresis.

Scenario 2 : $\alpha = 60\,\text{deg}$

In that case the dissipation β is higher. We can do the same experiment. If the angular velocity is $\Omega_M < \Omega_c$, the equilibrium is tilted. If the pendulum is slightly perturbed by hand, it goes back to equilibrium. If the perturbation is strong enough, the pendulum carries out one rotation and then stops at the stable equilibrium. This behavior is called excitability: a perturbation higher than a given threshold is necessary to "fire" the system. But there is no periodic behavior in this regime when $\Omega_M < \Omega_c$. The equilibrium reaches the value $\pi/2$ for $\Omega_M = \Omega_c$ and for a higher angular velocity, there is no equilibrium. The pendulum rotates but the regime is very different from the one we had with a weaker dissipation: the period is very long and the pendulum spends a long time near $\theta = \pi/2$.

IV. A POTENTIAL APPROACH

In this section, we will discuss the dynamics with a qualitative analog of the pendulum. The aim of this part is to give an intuitive understanding of the dynamics.

Let us write the equation of motion into the form

$$\theta'' + \nu\theta' = -\frac{\partial V(\theta)}{\partial \theta} \quad \text{with } V(\theta) = -\gamma\theta - \cos\theta \qquad (6)$$

We will describe the motion of a ball in the potential $V(\theta)$. The ball is acted upon by the weight of the ball and a viscous drag. The ball stays in contact with the potential, and does not jump. The motion of the ball is different from the motion of the pendulum because the kinetic energy has a different form. However the qualitative behavior is the same, and in particular the equilibrium positiion are the same. We will discuss the motion of the pendulum and give qualitative picture of the motion of the ball.

A. Periodic behavior and bistability

We start by considering the case of a non forced conservative pendulum. The potential is simply $V(\theta) = -\cos\theta$. The stable equilibrium is at $\theta = 0$ and the unstable one is at $\theta = \pi$ (figure 4). When one perturbs the stable solution the pendulum oscillates about its stable equilibrium. If one perturbs it strongly, it will oscillates about the nearest stable equilibrium solution. Now if one throws the pendulum it can go for ever. This corresponds to rotations. There is a separatrix between those two behavior: the trajectory joining two successive unstable points, which is called an homoclinic trajectory. This trajectory is a limit case: if one releases the pendulum from the unstable point, or infinitely close to the unstable point, it takes an infinite time to escape from the vicinity of this point, then it falls in the potential well and goes up again in an infinetely long time to the next unstable equilibrium.

When the external torque is slightly increased without dissipation (this is not realistic in the experiment) (figure 4), the two equilibria move but still exist. We call θ_s the stable equilibrium and θ_s the unstable one. Moreover we call θ_u the value of θ for which the potential V has exactly the value $V(\theta_u)$. If one perturbs slightly the stable equilibrium, and drops the pendulum with an angle between θ_b and θ_u and with no velocity, the pendulum oscillates about the stable solution. If the pendulum is released with an angle smaller than θ_b (see figure 4 D), it will overshoot the next potential maximum and will accelerate for ever. The first integral of equation (6) gives the speed increase between two successive unstable equilibria $\dot{\theta}_2^2 - \dot{\theta}_1^2 = 2\,(V(\theta_2) - V(\theta_1)) = 2\pi\gamma$. There is still an homoclinic trajectory serving as a separatrix between the oscillations and the rotations.

With weak dissipation, oscillations about θ_s are slightly damped. If one releases the pendulum at θ_b un-

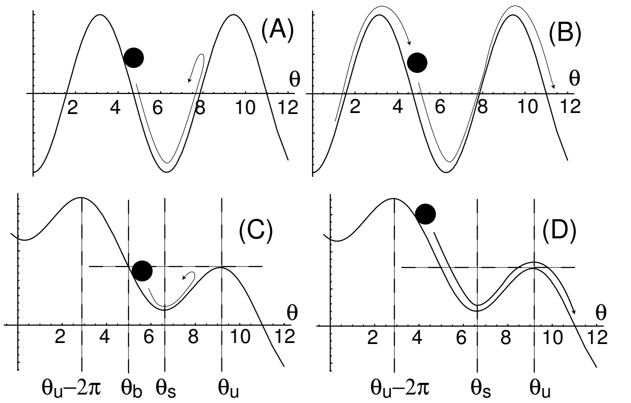

FIG. 4: Motion of a ball on a potential $V(\theta)$ without dissipation. The classical pendulum without forcing $\gamma = 0$ is above and the forced pendulum, $\gamma = 0.3$ is below.

like the case of a conservative pendulum, and because of damping, the ball will not reach θ_u. But if the dissipation is not too large, there is a point close to θ_b, we call it θ_l, such that if one releases the pendulum at θ_l, it will go asymptotically to θ_u. The point θ_l must verify an energetic balance between the energy dissipated because of damping and the potential energy variation. Hence multiplying equation by $\dot{\theta}$ and integrating once between $\tau = 0$ and $\tau = \infty$ yields

$$\beta \int_0^{\infty} (\theta')^2 dt = -V(\theta_u) + V(\theta_h) \tag{7}$$

We have used the initial and final conditions $\theta'|_{\tau=0} = 0$ and $\theta'|_{\tau=\infty} = 0$. This equation gives the energy balance but unfortunately can not be solved to find θ_l except if we know the explicit form of $\theta(\tau)$.

If the pendulum is released with an angle between $\theta_u - 2\pi$ and θ_l it will reach θ_u with a non zero velocity and then will continue to rotate. It will accelerate until it reaches (asymptotically) a periodic regime. The energetic balance verified by the periodic regime $\theta_p(\tau)$ is

$$\beta \int_0^{T} (\theta_p')^2 d\tau = 2\pi\gamma, \tag{8}$$

where T is the period of the rotations. If one throws the pendulum very fast, it will slow down until it reaches (asymptotically) the periodic regime.

FIG. 5: Motion of a ball on a potential $V(\theta)$ with dissipation. The pendulum with forcing $\gamma = 0.3$ and $\beta < \beta_0$ (A,B), the pendulum with $\gamma = 0.3$ and $\beta > \beta_0$ (C) and the pendulum with $\gamma > 1$ (D).

But if the damping is large, a pendulum dropped at $\theta_u - 2\pi$ will not reach the next unstable equilibrium. Hence there is no more oscillations. There is a critical damping $\beta_0(\gamma)$ for which $\theta_l = \theta_u - 2\pi$. The trajectory that connects two successive unstable points is called homoclinic. The condition if existence of is given by the balance of energy is

$$\beta \int_{-\infty}^{\infty} (\theta_h')^2 d\tau = 2\pi\gamma. \tag{9}$$

Below the value $\beta = \beta_0(\gamma)$ the pendulum can rotate and above this value the pendulum will stop on a stable equilibrium. This is the second bifurcation we wante to illustrate with our experiment. It is an homoclinic bifurcation, first studied by an Andronov.

B. Hysteresis

When the torque is large ($\gamma > 1$) there is no equilibrium (figure 5). In the reversible case, the pendulum rotates and accelerates for ever. If there is some dissipation, the pendulum will reach a periodic regime. An interesting point is that at the onset of the loss of equilibria, the behavior can be completely different depending on the damping rate.

If the damping is strong, this is the case of scenario 2, the periodic regime will be in some sense similar to the behavior before the disparition of equilibrium. The pendulum will slow down when it approaches $\theta = \pi/2$ and stay here a long time (details will come later) and then it will perform one rotation before coming back close to $\theta = \pi/2$. When γ decreases to a value below 1, the pendulum stops at its stable position close to $\pi/2$. There is no hysteresis in this regime.

When the damping is weak, this is the case of scenario 1, at the onset, the pendulum starts to rotates and accelerates until it reaches the periodic regime. Now this regime has a finite period. Actually the pendulum jumps to one periodic trajectory that was already there. When one decreases the forcing torque, even if the critical torque $\gamma = 1$ is crossed the pendulum keeps on rotating. When one keeps on decreasing the torque, the pendulum slows down close to the unstable equilibrium. There is a critical torque for which the the pendulum does not cross the unstable equilibrium and it goes back to the stable one.

Summary : There are two boundaries in the parameters space (γ, β) (figure 6). One of them is $\gamma = 1$ and it corresponds to the disparition of equilibria. The other one separates the domain where a periodic trajectory exists and the region where it doesn't exist. The shape of this curve $\beta = \beta_0(\gamma)$ has been obtained by a numerical experiment. There exist three regions in the parameter space that are separated by these boundaries. In region A the only stable solution is the stationnary solution. In the region B there are two stable solution a stationnary solution and a periodic rotation. In the region C the only stable solution is a periodic rotation.

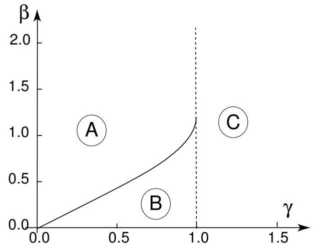

FIG. 6: Parameter space.

V. BEHAVIOR NEAR THE BIFURCATIONS

A. Loss of equilibria

Near the loss of equilibria, we write $\gamma = 1 + \epsilon$ where ϵ is small. In the regime $\epsilon < 0$, it is natural to write the angle as a development $\theta = \pi/2 + \theta_1 + ...$ The order of magnitude of the correction of the angle is $\theta_1 \sim \sqrt{|\epsilon|}$. It is simply deduced from the parabolic form of the curve $\sin\theta - 1$ near $\theta = \pi/2$. We substitute this ansatz in equation (2) and get at leading non zero order

$$\theta_1'' + \beta\theta_1' = \epsilon + \frac{1}{2}\theta_1^2 \tag{10}$$

The two equilibria are for $\epsilon < 0$, $\theta_1 = \pm\sqrt{-2\epsilon}$. Moreover equation (10) gives the time scale of the variation of the angle. If β is not small, the time scale of the variation of θ_1 is given by $\beta\theta_1' \sim \epsilon$ leading to a slow time scale $T = \sqrt{|\epsilon|}\tau$. The inertial term can then be dropped. Note that in the case $\beta = 0$ the time scale is $T = (|\epsilon|)^{1/4}\tau$.

It is worth remarking that this asymptotic development offers a description of the system near the point $\theta = \pi/2$. Consequently the periodic orbit for $\gamma > 1$ can not be described by this approach. But, when the dissipation is strong enough *i.e.* when the pendulum does not jump to an existing limit cycle, the pendulum spends a very long time close to the point $\theta = \pi/2$ and the time scale of the period of the rotating motion scales like $\sqrt{\epsilon}$. In the limit of a very dissipative system, the dynamic can be described analytically since we can drop the inertial term θ''. The equation of motion then reduces to

$$\theta' = \frac{1}{\beta}\left(\gamma - \sin\theta\right) \tag{11}$$

One can give explicit solution of this equation. For $\gamma > 1$, we have the explicit form of the periodic solution

$$\theta(\tau) = 2\arctan\left[\sqrt{\frac{\gamma^2-1}{\gamma^2}}\tan\left(\sqrt{\frac{\gamma^2-1}{\gamma^2}}\frac{\tau}{\beta}\right) + 1\right] \tag{12}$$

We plot the speed of the pendulum as a function of τ. It reveals that the pendulum spend a long time near $\theta = \pi/2$ (the plateau in the curve). The period of the oscillation is

$$T = \frac{\pi\gamma\beta}{\sqrt{\gamma^2-1}} \tag{13}$$

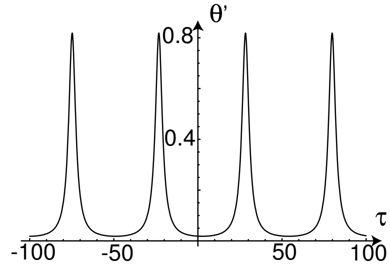

FIG. 7: Plot of the celerity $\theta'(\tau)$ as a function of time in the strong dissipation limit with $\gamma = 1.05$ and $\beta = 5$.

When γ is close to 1, the period scales like $\epsilon^{-1/2}$.

B. Loss of rotations

On figure 6, we have plotted the boundary $\beta = \beta_0(\gamma)$ of the region where rotations exist. A point of this curve corresponds to parameters where an homoclinic trajectory connecting two successive unstable point exists. There is a known situation for which this orbit exists : the pendulum without damping and without forcing. The homoclinic solution is then

$$\theta_h(\tau) = 2\arctan(\sinh\tau). \tag{14}$$

Now we perturb slightly the system with weak forcing and weak damping, and we study the persistence of that solution. We assume that β and γ are both of order ϵ and we perturb the solution θ_h and write

$$\theta = \theta_h(\tau) + \epsilon\theta_1(\tau) + \dots \tag{15}$$

In the initial equation (2), we use the classical procedure to get an energetic balance *i.e.* we multiply by θ' and integrate between τ_i and τ_f. We get

$$\frac{1}{2}\left[(\theta_f')^2 - (\theta_i')^2\right] + \beta\int_{\tau_i}^{\tau_f}(\theta')^2 dt - (\cos\theta_f - \cos\theta_i) = \gamma(\theta_f - \theta_i) \tag{16}$$

f (and i) subscripts denotes the values of the function at $\tau = \tau_f$ (and $\tau = \tau_i$). And we look for solution with $\tau_f = \infty$, $\tau_i = -\infty$, $\theta_f - \theta_i = 2\pi$ and $\theta_f' = \theta_i' = 0$. At order ϵ equation (16) reads

$$2\pi\gamma = \beta\int_{\tau_i}^{\tau_f}(\theta_h')^2 ds = 8\beta \tag{17}$$

Close to the homoclinic bifurcation *i.e.* when $\beta \to \beta_0^-$, the period of oscillation diverges. When $\beta = \beta_0$ the orbit

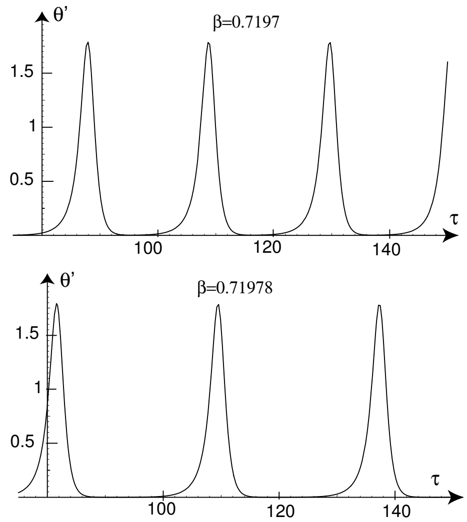

FIG. 8: Plot of the celerity $\theta'(\tau)$ as a function of time close to the homoclinic curve with $\gamma = 0.8$ for two values of β.

is $\theta_h(\tau)$ and it verifies the energy balance of equation 9 while when it is periodic it verifies equation 8.

The determination of the period of rotations is based on the following argument : when β decreases, $\beta = \beta_0(1-\epsilon)$, the energy lost by viscous damping between $\theta_u - 2\pi$ and θ_u decreases. To compensate this excess of energy, the period decreases. Moreover for large period, the pattern between $-T/2$ and $T/2$ is in first approximation θ_h (figure 8). Then the energy balance reads

$$-\beta_c\epsilon\int_{-\infty}^{\infty}(\theta_h')^2d\tau = \beta_c\int_{-T/2}^{T/2}(\theta_h')^2d\tau - \beta_c\int_{-\infty}^{\infty}(\theta_h')^2d\tau \tag{18}$$

or

$$-\epsilon 2\pi\gamma \approx \beta_c\int_{-\infty}^{-T/2}(\theta_h')^2d\tau - \beta_c\int_{T/2}^{\infty}(\theta_h')^2d\tau \tag{19}$$

To compute the integrals of equation 19, the behavior of θ_h near the unstable point is approximate by the linearization of equation 2 near these two points. The linearized equation is

$$\theta'' + \beta\theta' + (\cos\theta_u)\theta = 0, \tag{20}$$

where $\cos\theta_u = -(1-\gamma^2)^{1/2}$ is negative. The two characteristic exponents that measure the convergence (respectively divergence) to (respectively from) the unstable point are a_- (respectively a_+) given by

$$a_\pm = \frac{\beta}{2}(-1 \pm \sqrt{1 \pm 4\frac{\cos\theta_u}{\beta}}), \tag{21}$$

and close to $+\infty$ (respectively $-\infty$) θ_h' behaves like e^{a_τ} (respectively e^{a_τ}). These two behaviors are introduced in equation 19 yielding

$$-\epsilon 2\pi\gamma \approx \mathrm{e}^{-a_+T} - \mathrm{e}^{a_-T} \tag{22}$$

Which gives a divergence of the period

$$T \sim a_- \log\epsilon. \tag{23}$$

VI. SUMMARY AND DISCUSSION

In this paper, a simple mechanical device was studied both experimentally and theoretically. The dynamics depends on two parameters: one measures the damping and the other one the forcing. The experiment as well as the qualitative analysis reveals the existence of two types of bifurcations: a local bifurcation, the loss of equilibria and a global bifurcation, the homoclinic bifurcation.

The loss of equilibria occurs for a given value of forcing. If the damping is weak, the bifurcation is subcritical and the pendulum rotates with a finite period. On the other hand, if the damping is strong, the bifurcation is supercritical and the period of rotations just above threshold scales like $\sqrt{\epsilon}$ where ϵ is the distance from threshold. For a given torque, if the damping is not too strong, stable periodic rotations coexist with equilibria. These periodic rotations arise through an homoclinic bifurcation at $\beta = \beta_0(\gamma)$. The exact position of the curve $\beta_0(\gamma)$ in parameter space can not be determined explicitely. The periodic orbit arising through this bifurcation has a diverging period as $T \sim -\ln\epsilon$ where ϵ is the distance from threshold.

The qualitative analysis developped here applies to more physical situations. For example, a model of sandpiles[4], exhibits the bifurcations described of our simple experiment. The two bifurcations there correponds to the static angle where the sandpile flows and the dynamical angle (our homoclinic bifurcation) where a finite perturbation leads to avalanches.

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tures on physics, vol. 2 (Addision-Wesley, Reading, 1963).
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[5] $\pi - \theta_s$ is an unstable equilibrium position as we will see further